Calculus I Test

Instructions. It is in your best interest to take this exam by yourself, without external resources other than a graphing calculator, so that you are placed into the appropriate first mathematics course at St. Olaf. Write out your answers on paper (don’t do the problems in your head), then check your answers with the solutions that start on page 3. There is nothing to be gained from looking at the solutions first, so please abide by this request for honesty.

You should score (perhaps with a small amount of review) at least 80% (16 out of 20 questions) on this sample exam in order to skip Calculus I (Math 120) and go straight into Calculus II (Math 126). If you score less than 80% on the exam and fewer than 4 out of the first 7 algebra questions, you are advised to take Calculus I with Review (Math 119). See page 3 for solutions.

1. Simplify as much as possible: \( \frac{x^5(x^2y)^3}{y^2} \).

2. Express as a single fraction containing no parentheses: \( \frac{3}{z+6} + \frac{2}{z+5} \).

3. Solve for \( b \): \( \log_{10}(5b) = 2 \).

4. Some values of a function \( g \) are shown in the table below. If \( g \) is a linear function, what is the value of \( c \)?

<table>
<thead>
<tr>
<th>( t )</th>
<th>( y(t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>8</td>
</tr>
<tr>
<td>5</td>
<td>( c )</td>
</tr>
</tbody>
</table>

5. At what points does the graph of the function \( q(x) = \frac{x^2 + 3x}{x - 4} \) cross the \( x \)-axis?

6. If \( (1,7) \) is a point on the graph of \( p(x) = x^3 - 2x^2 + 4x + d \), what is the value of \( d \)?

7. If \( f \) is the function \( f(x) = 3x^2 + x - 7 \), evaluate \( f(1+h) \). (Simplify your answer so that it does not contain parentheses.)

8. Find the derivative of \( y = (2x^4 + 7x)^6 \).

9. Find the derivative of \( y = x \sin(x) \).

10. Find the derivative of \( y = e^{x^2+4x} \).

For problems 11-13, consider the function \( f(x) = 2x^3 - 9x^2 + 12x + 1 \) and its first and second derivatives \( f'(x) = 6x^2 - 18x + 12 \) and \( f''(x) = 12x - 18 \).

11. Find all intervals on which \( f \) is increasing.

12. Find all intervals on which \( f \) is concave up.

13. Find all local maxima and minima for \( f \).

For problems 14-16, refer to the graph of \( f' \) shown in Figure 1.

14. On which of the following open intervals is \( f \) decreasing? (Note: this question is about the function \( f \), but the graph is of \( f' \).)

   - \((0,1.28), \ (1.28,1.78), \ (1.78,2.18), \ (2.18,2.50), \ (2.50,2.80), \ (2.80,3.08), \ (3.08,3.15)\)
15. On which of the following open intervals is \( f \) concave down?

\((0, 1.28), \ (1.28, 1.78), \ (1.78, 2.18), \ (2.18, 2.50), \ (2.50, 2.80), \ (2.80, 3.08), \ (3.08, 3.15)\)

16. At which of the following points does \( f \) reach a local maximum?

\(0, \ 1.28, \ 1.78, \ 2.18, \ 2.50, \ 2.80, \ 3.08, \ 3.15\)

![Figure 1: The graph of \( f' \), the derivative of \( f \)](image)

17. Consider the data shown below for a function \( f \). Estimate the instantaneous rate of change of \( f \) at \( x = 3.5 \) (there is more than one way to do this).

<table>
<thead>
<tr>
<th>( x )</th>
<th>0.0</th>
<th>1.0</th>
<th>1.7</th>
<th>2.3</th>
<th>2.5</th>
<th>3.5</th>
<th>4.2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>2.0</td>
<td>3.7</td>
<td>5.0</td>
<td>6.4</td>
<td>7.7</td>
<td>8.9</td>
<td>10.0</td>
</tr>
</tbody>
</table>

18. The function \( g(x) \) satisfies \( g(3) = 10 \) and \( g'(3) = 2 \). Estimate the value of \( g(3.1) \).

19. Find the equation of the line tangent to \( f(x) = 3x^3 + \ln(x) \) at \( x = 1 \).

20. Evaluate \( \int_1^2 (x^3 + 2x) \, dx \).
**Solutions to Calculus I Test**

**Instructions:** Compare your solutions with the answers given below. Each of the 20 questions is worth 1 point, but you can give yourself 1/2 point of partial credit if appropriate.

When you are done grading your test, total the points. A score of 16 or more suggests that you are ready for Math 126, Calculus II. If you score below 16, we recommend that you take Math 120, Calculus I.

1. \( \frac{x^5 (x^2y)^3}{y^2} = \frac{x^5x^6y^3}{y^2} = x^{11}y \)

2. \( \frac{3}{z+6} + \frac{z}{z+5} = \frac{3(z+5) + z(z+6)}{(z+6)(z+5)} = \frac{z^2 + 9z + 15}{z^2 + 11z + 30} \)

3. Since \( \log_{10}(5b) = 2 \) implies \( 10^2 = 5b \), \( b = 20 \).

4. A linear function has constant rate of change (slope) so \( \frac{8 - 3}{2 - (-1)} = \frac{c - 8}{5 - 2} \) Hence \( c = 13 \).

5. At the points where the graph crosses the \( x \)-axis \( q(x) = 0 \). This implies \( 0 = x^2 + 3x = x(x + 3) \), so \( x = 0 \) or \( x = -3 \).

6. Since \( (1, 7) \) is on the graph, \( p(1) = 7 \). Hence \( 7 = p(1) = 1 - 2(1) + 4(1) + d \), which implies \( d = 4 \).

7. \( f(1 + h) = 3(1 + h)^2 + (1 + h) - 7 = 3(1 + 2h + h^2) + 8 - 6 = 3h^2 + 7h - 3 \)

8. \( y' = 6(2x^4 + 7x)^5(8x^3 + 7) \)

9. \( y' = x\cos(x) + \sin(x) \)

10. \( y' = (2x + 4)e^{x^2 + 4x} \)

11. \( f'(x) = 6x^2 - 18x + 12 = 6(x^2 - 3x + 2) = 6(x - 2)(x - 1) \). We know that \( f \) is increasing when \( f' \) is positive. This occurs on the intervals \(( -\infty, 1) \) and \(( 2, \infty) \).

12. \( f''(x) = 12x - 18 = 6(2x - 3) \). We know that \( f \) is concave up when \( f'' \) is positive. This occurs on the interval \(( 3/2, \infty) \).

13. The critical points for \( f \) are where \( f'(x) = 0 \), so at \( x = 1 \) and \( x = 2 \). The point \((1, f(1)) = (1, 6) \) is a local maximum while \((2, f(2)) = (2, 5) \) is a local minimum.

14. \( f \) is decreasing when \( f' \) is negative, so on the intervals \(( 1.78, 2.18 ) \), \(( 2.18, 2.50 ) \), and \(( 3.08, 3.15 ) \).

15. \( f \) is concave down when \( f'' \) is negative. In turn, \( f'' \) is negative when \( f' \) is decreasing, so on the intervals \(( 1.28, 1.78 ) \), \(( 1.78, 2.18 ) \), \(( 2.80, 3.08 ) \), and \(( 3.08, 3.15 ) \).

16. \( f \) reaches a local maximum when \( f' \) changes from positive to negative, so at \( x = 1.78 \) and \( x = 3.08 \).

17. A good way to estimate the instantaneous rate of change at \( x = 3.5 \) is to use the nearest data point: \( \frac{10 - 8.9}{1.51 - 1.5} \approx 1.571 \). Alternative answers are \( \frac{8.9 - 7.7}{1.5 + 1.2} \approx 1.386 \).

18. \( g(3.1) \approx 10 + 2(0.1) = 10.2 \)

19. The slope of the tangent line is given by \( f'(1) \). We see that \( f'(x) = 9x^2 + \frac{1}{2} \), so \( f'(1) = 10 \). The point of tangency is \((1, f(1)) = (1, 3) \). Finally, using the point-slope formula we get that the tangent line has equation \( y - 3 = 10(x - 1) \), or \( y = 10x - 7 \).

20. \( f_1^2 (x^3 + 2x) \) \( dx = \frac{[x^4 + x^2]_1}{1} = \left( \frac{16}{4} + 4 \right) - \left( \frac{1}{4} + 1 \right) = 6.75 \)

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